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Approximate equilibria of Blotto-type games

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Résumé. The Colonel Blotto game is a well-known strategic resource allocation problem with important applications in a vast range of domains, from advertising to security. Two players simultaneously allocate a fixed budget of resources across n battlefields, each player trying to maximize the aggregate value of the battlefields where she has higher allocation. Despite its long-standing history and importance, the Colonel Blotto game still lacks a complete Nash equilibrium solution in its general form with asymmetric players' budgets and heterogeneous battlefield values.
In this paper, we propose an approximate equilibrium for this general case. We construct a simple strategy (the *independently uniform* strategy) and prove that it is an epsilon-equilibrium. We give a theoretical bound on the approximation error in terms of the number of battlefields and players' budgets which identifies precisely the parameters regime for which our strategy is a good approximation. We also investigate an extension to the discrete version (where players can only have integer allocations), for which we proposed a modified strategy that can be computed very efficiently and gives an epsilon-equilibrium of the game.

Mots-clefs : game theory, resource allocation, Colonel Blotto game, epsilon-equilibrium

1 Introduction and Game Formulation

The Colonel Blotto game is one of the first-examined problems in modern game theory, yet it remains as a challenge to completely characterize the equilibrium of its general form. However, interestingly and surprisingly, the game's formulation is very simple. Two players, denoted A and B, have fixed budgets of resources X^A and $X^B \geq X^A$, respectively. Players simultaneously allocate their resources across n battlefields, where each battlefield $i \in \{1, 2, \ldots, n\}$ has an arbitrary value v_i. We denote by $V_n := \sum_{i=1}^n v_i$ the total value of all battlefields. A pure strategy of player $p \in \{A, B\}$ is a vector $\boldsymbol{x}^p = (x_i^p)_{i=1,2,\ldots n} \in [0, X^p]^n$ which holds the budget constraint $\sum_{i=1}^n x_i^p \leq X^p$. Each player wins the battlefields on which he allocated more resources than the opponent[1] and receives a utility equal to the aggregate value of the battlefields he won.

[1] We consider a generic tie-breaking rule where player A gains αv_i and player B gains $(1-\alpha)v_i$ if $x_i^A = x_i^B$, with an arbitrary $\alpha \in [0, 1]$.

The Colonel Blotto game was first introduced in 1921 [3]. In 1950, a solution (complete characterization of the Nash equilibrium) was given for the case of symmetric players' budgets [4]. A solution for the case of asymmetric players' budgets (i.e., $X^A \neq X^B$) was then given in 2006 [5], but only for homogeneous battlefields (i.e., $v_i = v_j, \forall i, j$).

2 The Independently Uniform strategy as an εV_n-equilibrium

In [4, 5], the challenge in equilibrium analysis of the Colonel Blotto game lies in constructing a joint distribution both yielding specific marginal distributions and satisfying the budget constraints. Inspired by these works, we define for each battlefield i the marginal distributions

$$F_{A_i^*}(x) := \left(1 - \frac{X^A}{X^B}\right) + \frac{x}{2\frac{v_i}{V_n}X^B}\frac{X^A}{X^B}, \text{ if } x \in \left[0, 2\frac{v_i}{V_n}X^B\right], \tag{1}$$

$$F_{B_i^*}(x) := \frac{x}{2\frac{v_i}{V_n}X^B}, \text{ if } x \in \left[0, 2\frac{v_i}{V_n}X^B\right]. \tag{2}$$

Intuitively, $F_{B_i^*}$ is the (continuous) uniform distribution on $\left[0, 2v_iX^B/V_n\right]$ and $F_{A_i^*}$ is the distribution placing a positive mass $\left(1 - X^A/X^B\right)$ at 0 and uniformly distributing the remaining mass on $\left(0, 2v_iX^B/V_n\right]$. It is easy to prove that, if there exists a feasible strategy (respecting the budget constraints) by which the marginal allocation at battlefield i of player p is $F_{p_i^*}$, this strategy is an equilibrium of the game. The construction of such a strategy (as well as its existence), however, is still a challenging open question. In this paper, we take a different perspective: we propose a simple strategy based on independently drawing the allocations from $F_{A_i^*}$ and $F_{B_i^*}$ and then re-scaling them to ensure the budget constraints. This is clearly a feasible strategy, but it no longer has the marginals $F_{A_i^*}$ and $F_{B_i^*}$. We investigate the question of whether this strategy is an approximate equilibrium, and how the approximation quality depends on the game's parameters. We first define the strategy formally.

Definition 1 *First, let the variables A_k^* and B_k^* be drawn independently from distributions $F_{A_k^*}$ and $F_{B_k^*}$ for $k = 1, \cdots, n$. Then, the Independently Uniform (IU) strategy is the mixed strategy defined as follows: it is the joint distribution of the random variables $(A_i^n)_i$ and $(B_i^n)_i$ defined for all i by $A_i^n := \frac{A_i^*}{\sum_{k=1}^n A_k^*}X^A$ and $B_i^n := \frac{B_i^*}{\sum_{k=1}^n B_k^*}X^B$.*

Note that the summation $\sum_{k=1}^n B_k^*$ equals 0 with probability zero, while $\sum_{k=1}^n A_k^*$ equals 0 with a positive probability, but this probability vanishes as n grows. Finally, we can prove that for any $\varepsilon > 0$, if n is large enough, no player has a unilateral deviation from (IU) strategy that can benefit him more than εV_n, a small fraction of the total payoff. In other words, the (IU) strategy is an εV_n-equilibrium of the game. Moreover, we can precisely bound the approximation error ε in terms of number of battlefields n.

Theorem 2 *The (IU) strategy is an εV_n-equilibrium of the Colonel Blotto game with n battlefields and total value V_n, where $\varepsilon \leq \tilde{\mathcal{O}}\left(n^{-\frac{1}{2}}\right)$.*

Although the idea behind the IU strategy is simple, the proof of Theorem 2 is not straightforward. Intuitively, since marginals $F_{A_i^n}$ and $F_{B_i^n}$, induced from the IU strategy, cannot be explicitly expressed, we need to evaluate them via other terms. This can be done by using a concentration inequality to prove that $F_{A_i^n}$ and $F_{B_i^n}$ uniformly converge towards $F_{A_i^*}$ and

$F_{B_i^*}$, respectively. Using the special properties of distributions $F_{A_i^*}$ and $F_{B_i^*}$ together with the portmanteau lemma, we can analyze and approximate (with arbitrary small error $\varepsilon > 0$) the payoffs given by $F_{A_i^n}$ and $F_{B_i^n}$, which concludes the theorem. A more difficult task is to carefully analyze the convergence rate of all of the above analysis which turns out to be $n \geq \mathcal{O}\left(\varepsilon^{-2}\ln(\varepsilon^{-1})\right)$, hence $\varepsilon \leq \tilde{\mathcal{O}}\left(n^{-\frac{1}{2}}\right)$.

3 Extensions to the discrete Colonel Blotto game

There are several practical applications of the Colonel Blotto game involving indivisible resources which motivate the study of the discrete version of the game where the parameters X^A, X^B, and all of the players' allocations are required to be integers. In this discrete game, each player strategy-set is finite, but its cardinal grows exponentially as a function of the game parameters. Recently, [1] and [2] proposed two algorithms to compute the exact Nash-equilibrium of the discrete Colonel Blotto game. However, these works only find the equilibrium marginals and have an inefficient running time, even on games with medium-scale parameters. The cases where these algorithms are computationally infeasible are exactly those where our work can contribute. With a slight modification of the (IU) strategy (basically, a smart rounding process to ensure integer allocations), we design an extremely efficient algorithm to construct an εV_n-equilibrium of the discrete game, given that the number of battlefields and players' budgets are large. The rounding process requires a fine-grained discretization of the game, achieved when the ratios X^A/n and X^B/n are large. Moreover, we can numerically evaluate ε by using another algorithm which compares any player's payoff in the εV_n-equilibrium and the optimal payoff he gets from playing pure strategy against the (empirical) equilibrium marginals of the opponent. This polynomial-time algorithm, based on Dynamic Programming techniques, has its own stand-alone significance, where the scope of applicability goes beyond this result and it can work against any given set of the opponent's marginals. Our algorithms are implemented in several experimental simulations which show a noteworthy trade-off between running time and accuracy of the algorithm compared to the algorithms in [1] and [2].[2]

Références

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[2]For example, our algorithm determines a 0.02-equilibrium for games with $n = 150, X^A = 8000, X^B = 9600$ in less than 2 hours, while [2]'s algorithm takes more than one day to find the exact NE of games with n, X^A, X^B around 50.