Journées SMAI MODE 2018

ON THE EXISTENCE OF SUBGAME PERFECT EQUILIBRIA IN DISCONTINUOUS PERFECT INFORMATION GAMES

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Résumé: In this paper, we prove that for a large class of discontinuous perfect information games, called extensive-form better-reply secure, a subgame perfect equilibrium exists. This is the counterpart of Reny's existence theorem (true for normal form games) for extensive form games. Roughly, a game is extensive-form better-reply secure for every strategy profile s^* which is not a subgame perfect equilibrium, there is a deviation of some player i which gives him a payoff strictly above u_i^*, even if one perturbs slightly the path defined by s^* after i.

Mots-clefs : discontinuous game, better-reply security, extensive-form game.

1 Introduction

It is well known that the existence of a subgame perfect equilibrium (SPE) in pure strategies can be obtained through a standard backward induction scheme, in every extensive-form game with: Perfect information [1], a finite horizon and a finite number of actions at each period.
Under some continuity assumption on the payoffs, Fudenberg and Levine [2] have proved that the existence proof can be extended to infinite horizon games. But as shown in Harris [3], their argument can not be extended to the case of an infinite numbers of actions at each period.
Harris [3] (then Reny, Hellwig and Robson [4] and more recently Carmona [5]) have provided existence proofs of a SPE for infinite sets of actions at each period, when the payoff functions are assumed to be continuous.
Since then, no general existence proof has been given, allowing some kind of discontinuities of payoffs. Yet, a large literature on games with discontinuous payoffs has developed for more than 20 years, but all the attention has been concentrated on normal form games. One of the most used and analyzed result is Reny's theorem, which proves the existence of a pure Nash equilibrium for normal form games with possibly discontinuous payoffs. The class of games considered by Reny is called better-reply secure games, and encompasses many standard examples in the literature, as auctions, location games, or duopoly games.
In this paper, we provide an existence result of a SPE for a class of discontinuous extensive-form games. This class is called extensive-form better-reply secure, and has some similarity, in terms of definition, with the class of extensive-form better-reply secure games. But the connection between normal form games and extensive form-games is unformal, since none of the two models implies the other, and in fact, our method of proof is unrelated to Reny's one.

2 THE MODEL

We consider an extensive-form game with $N \geq 2$ players $i = 1, ..., N$. Each player i has an action set X_i, assumed to be a topological space and a (possibly discontinuous) payoff function $u_i : X_1 \times ... \times X_N \to \mathbf{R}$.
The game can be formalized by the following directed tree Γ: the root is denoted x_0, and represents

the moment just before player 1's action; the set of nodes is $V = \cup_{i=0}^{N} H_i$, where for every $i = 0, ..., N$,

$$H_i = \{(x_0, ..., x_i) \in X_0 \times X_1 \times X_2 \times ... \times X_i : \forall j = 1, ..., i, x_j \in X_j(x_0, ..., x_{j-1})\},$$

where $X_j(x_0, ..., x_{j-1}) \subset X_j$ is the set of admissible actions of player j given the previous actions $(x_0, x_1, ..., x_{j-1})$ which have been played.
The set H_i is called the set of *histories for player* $i+1$, $i = 0, ..., N-1$. In particular, the set of terminal nodes is H_N.

Assumptions
(A1) The payoff functions $u_i : X_1 \times ... \times X_N \to \mathbf{R}$ are assumed to be *bounded*, $i = 1, ..., N$.
(A2) For every $j = 1, ..., N$, X_j is a compact set, and $\{(x_1, ..., x_{j-1}, x_j) : \forall j = 1, ..., i,\ x_j \in X_j(x_0, ..., x_{j-1})\}$ is closed in $X_1 \times ... \times X_j$.

Throughout this paper, $X_1 \times ... \times X_N$ will be endowed with the product topology, and each $H_i \subset X_1 \times ... \times X_i$ is endowed with the induced topology. In particular, under assuption $(A2)$, each H_i is compact.
Now, it is convenient to associate to the extensive-form game above the following normal form game denoted G:

- The players of G are $i = 1, ..., N$.
- A strategy s_i of player $i \in N$ is a mapping from $H_{i-1} \to X_i$ which associates to every possible history $h_{i-1} \in H_{i-1}$ for player i some action $s_i(h_{i-1}) \in X_i(h_{i-1})$. In particular, a strategy s_i for player $i = 1$ can also be equivalently described as an element $x_1 \in X_1$ (an equivalent description that we will used in this paper). For every $i \in N$, let S_i be the set of strategies of player i. To define the payoff functions of G, fix a strategy profile $s = (s_1, ..., s_N) \in S := S_1 \times ... \times S_N$. This generates a *path* in the tree Γ, denoted $p(s)$, which is a sequence of decisions, $(x_1, ..., x_N)$ defined inductively by $x_1 = s_1, x_2 = s_2(x_1), ..., x_N = s_N(x_1, ..., x_{N-1})$.
- Last, the payoff of player $i \in \{1, ..., N\}$ in G is defined by

  $$U_i(s_1, ..., s_N) := u_i(p(s)).$$

  Each set S_i can be identified to $X_i^{H_{i-1}}$, and it is endowed with the product topology (and thus is compact, since X_i is compact). The set $S = S_1 \times ... \times S_N$ of strategy profiles is also endowed with product topology (and thus is compact).

Similarly, for every $i \in N$ and for every history $h_{i-1} = (x_1, ..., x_{i-1}) \in H_{i-1}$, a normal form game *played after* h_{i-1}, denoted $G(h_{i-1})$, can be defined as follows: the players' set is $\{i, i+1, ..., N\}$. A strategy s_j of player $j \in N$ is, again, a mapping from $H_{j-1} \to X_j$ (but in fact, only the values of $s_j(h_{i-1}, .)$ will be relevant for the definition of the payoffs in $G(h_{i-1})$). Every strategy profile $s = (s_1, ..., s_N)$ defines a unique path beginning at node $h_{i-1} = (x_1, ..., x_{i-1})$ and finishing at a terminal node, denoted $p_{|h_{i-1}}(s)$ (here, only the information of $s_i, .., s_N$ in s is relevant to be able to define $p_{|h_{i-1}}(s)$), as follows:
$x_i = s_i(x_1, ..., x_{i-1}), x_{i+1} = s_{i+1}(x_1, ..., x_{i-1}, x_i), ..., x_N = s_N(x_1, ..., x_{N-1})$.
Finally, at every $s = (s_1, ..., s_N)$, we can define a payoff in $G(h_{i-1})$ for each player $j = i, ..., N$, denoted $U_{j|h_{i-1}}(s_i, ..., s_N)$, by $U_{j|h_{i-1}}(s_i, ..., s_N) = u_j(h_{i-1}, p_{|h_{i-1}}(s))$. The game $G(h_{i-1})$ is called the subgame of G beginning at h_{i-1} (in particular $G(\{x_0\}) = G$).
Throughout this paper, we shall denote $\Gamma = ((X_i)_{i\in N}, (u_i)_{i\in N})$ the pair of action sets and payoff functions, defining the extensive-form game. The associated normal game will be denoted $G = ((S_i)_{i\in N}, (U_i)_{i\in N})$.

We now define the standard equilibrium notion associated to such game G.

Definition 2.1 A strategy profile $s = (s_1, ..., s_N)$ is a subgame-perfect equilibrium (SPE) of $\Gamma = ((X_i)_{i\in N}, (u_i)_{i\in N})$ if for every $i \in N$ and every history $h = (x_0, ..., x_{i-1})$, $(s_i, s_{i+1}, ..., s_N)$ is a Nash equilibrium of $G(h)$.

Equivalently, $s = (s_1, ..., s_N)$ is a subgame-perfect equilibrium (SPE) of Γ if for every $i = 1, ..., N$, for every history $h_{i-1} = (x_0, ..., x_{i-1}) \in H_{i-1}$ and every deviation d_i of every player i,

$$u_i(h_{i-1}, d_i, p_{|(h_{i-1},d_i)}(s_{i+1}, .., s_N)) \leq u_i(h_{i-1}, p_{|h_{i-1}}(s_i, .., s_N)).$$

3 Extensive-form Better-reply security

Definition 3.1 Player $i \in N$ can secure a payoff $a \in \mathbf{R}$ at $h_{i-1} \in H_{i-1}$, given $s \in S$, if there exists a strategy $d_i \in X_i(h_{i-1})$ such that $u_i(h_{i-1}, d_i, x') \geq a$ for every x' in some neighborhood of $p_{|(h_{i-1},d_i)}(s)$.

Definition 3.2 A vector $u^* \in \mathbf{R}^N$ is a limit vector of the profile of payoff functions $u = (u_1, ..., u_N)$ at $s^* \in S$ given $h_{i-1} \in H_{i-1}$, if

$$(h_{i-1}, p_{|h_{i-1}}(s^*), u^*) \in \overline{\{(x, u(x)) : x \in H_N : (x_1, ..., x_{i-1}) = h_{i-1}\}}.$$

Definition 3.3 An extensive-form game $G = (X, u)$ is *extensive-form better-reply secure (e.b.r.s.)* is for every strategy profile s^* which is not a SPE , there exists $k \geq 1$ and $h_{k-1} \in H_{k-1}$ such that: for every limit vector of u^* at $s^* \in S$ given $h_{k-1} \in H_{k-1}$, player k can secure a payoff strictly above u^*_k given $h_{k-1} \in H_{k-1}$.

Theorem 3.1 Under Assumptions (A1) and (A2), for every extensive-form better-reply secure game G, there exists some SPE.

Références

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