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Fast Computation of Equilibria in Monotone Routing Games and Application to Electricity Demand Response

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Résumé. Routing games find many applications in various fields such as transportation, telecommunications or energy. The set of Nash Equilibria in this class of games is in general hard to compute. Here, we focus on the particular case of a parallel network and we address the computational issue of equilibria by providing two algorithms: the cycling best response dynamics and a projected gradient descent method. Under some monotonicity assumptions, we prove the convergence of those methods and we provide an upper bound on their convergence rate. Our convergence results state that, using one of these algorithms, the unique equilibrium of the game can be computed at an arbitrary precision in polynomial time. We give a practical application in the energy sector, where this framework and the associated results can be used to optimize the electricity consumption of flexible users.

Mots-clefs : Congestion Game, Best Response, Nash Equilibrium, Demand Response.

We focus on a particular class of N-person games, splittable routing congestion games on a parallel network. We expose the results given in our recent paper [2] in a more general setting. A routing congestion game on a parallel network (see [4]) is defined as a set of players $\mathcal{N} = \{1, \dots, N\}$, a set of edges $\mathcal{T} = \{1, \dots, T\}$, a tuple of **strictly increasing**, **convex** and differentiable cost functions $c_t : \mathbb{R} \to \mathbb{R}$ for $t \in \mathcal{T}$ and, for each player i, a feasibility (strategies) set $\mathcal{X}_i$ that we assumed to be **convex** and **compact**. Given a profile $(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$, the cost of each player i is given by:

$$b_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) \stackrel{\text{def}}{=} \sum_{t \in \mathcal{T}} x_{i,t} c_t(x_t) \ , \tag{1}$$

where $x_t = \sum_{i \in \mathcal{N}} x_{i,t}$ is the aggregated load on edge t.
The basic assumptions above are sufficient to ensure the existence of a Nash Equilibrium. However, computing the Nash Equilibria of a N-person game is a hard problem: unlike classical congestion games, the existence of a potential function [3] for a generic instance of atomic splittable congestion game is unknown. Here, we provide some conditions under which two simple algorithms — the cycling best response and a simultaneous projected gradient descent — compute a Nash Equilibrium to an arbitrary precision in polynomial time.

Two Algorithms and their Convergence to the Nash Equilibrium. One of the most natural way to compute an equilibrium is to compute alternating minimization of b_i for each player i . For a given profile $\boldsymbol{x} = (\boldsymbol{x}_i, \boldsymbol{x}_{-i})$, we denote by BR_i the *Best Response* function of player i: $\mathrm{BR}_i : \boldsymbol{s}_i \mapsto \arg\min_{\boldsymbol{x}_i \in \mathcal{X}_i} \sum_t x_{i,t} c_t(s_{i,t} + x_{i,t})$, which depends only of the sum of the load of others players $\boldsymbol{s}_i \stackrel{\text{def}}{=} \sum_{j \neq i} \boldsymbol{x}_j$. This leads to the following algorithm:

Algorithm 1 Cycling Best Response Dynamics (CBRD)

Require: $\boldsymbol{x}^{(0)},\ k_{\max},\ \varepsilon_{\text{stop}}$

$k \leftarrow 0,\ \varepsilon^{(0)} \leftarrow \varepsilon_{\text{stop}}$

while $\varepsilon^{(k)} \geq \varepsilon_{\text{stop}}\ \ \&\ \ k \leq k_{\max}$ **do**

  for $i = 1$ to N **do**

    $s_i^{(k)} = \sum_{j<i} \boldsymbol{x}_j^{(k+1)} + \sum_{j>i} \boldsymbol{x}_j^{(k)}$

    $\boldsymbol{x}_i^{(k+1)} \leftarrow \text{BR}_i(s_i^{(k)})$

  end for

  $\varepsilon^{(k)} = \left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\|$

  $k \leftarrow k + 1$

end while

In this algorithm, coordinates are updated in a round-robin fashion as in the Gauss-Seidel method. Another approach to compute an equilibrium is to perform a projected gradient descent, considering the gradient of each objective function b_i, which gives the below algorithm:

Algorithm 2 Simultaneous Improving Response Dynamics (SIRD)

Require: $\boldsymbol{x}^{(0)},\ k_{\max},\ \varepsilon_{\text{stop}},\ \gamma$

$k \leftarrow 0,\ \varepsilon^{(0)} \leftarrow \varepsilon_{\text{stop}}$

while $\varepsilon^{(k)} \geq \varepsilon_{\text{stop}}\ \ \&\ \ k \leq k_{\max}$ **do**

  for $n = 1$ to N **do**

    $\boldsymbol{x}_n^{(k+1)} \leftarrow \Pi_{\mathcal{X}_n}\left(\boldsymbol{x}_n^{(k)} - \gamma \nabla_n b_n(\boldsymbol{x}_n^{(k)}, \boldsymbol{x}_{-n}^{(k)})\right)$

  end for

  $\varepsilon^{(k)} = \left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\|$

  $k \leftarrow k + 1$

end while

In general, those two algorithms may not converge. To prove the convergence in our case, we introduce the notion of *strong stability* defined below.

Definition 1 ***Stable Game.***
A minimization game $\mathcal{G} = (\mathcal{N}, \mathcal{X}, (b_i)_i)$ is A-strongly stable, with a constant $A > 0$, iff:

$$\forall \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X},\ (\boldsymbol{x}' - \boldsymbol{x})^T . \left(F(\boldsymbol{x}') - F(\boldsymbol{x})\right) \geq A \left\|\boldsymbol{x} - \boldsymbol{x}'\right\|^2 \ , \tag{2}$$

where $F = (\nabla_{\boldsymbol{x}_i} b_i)_{i\in\mathcal{N}}$. If $A \geq 0$, we say that $\mathcal{G}$ is stable.

Stability is sufficient to ensure the uniqueness of Nash Equilibrium in the game $\mathcal{G}$ [5]. It is also a sufficient condition to ensure the convergence of Algorithm 2, as stated below:

Theorem 1 *Denote by L_n a Lipschitz constant of $\nabla_n b_n$ and $L \stackrel{\text{def}}{=} \max_n L_n$. If the game is A-strongly stable, SIRD converges with step $\gamma \leq \gamma^* \stackrel{\text{def}}{=} A/(NL^2)$. Moreover, for $\gamma = \gamma^*$, we have:*

$$\left\|\boldsymbol{x}^{\text{NE}} - \boldsymbol{x}^{(k)}\right\|_2 \leq \eta^k \left\|\boldsymbol{x}^{\text{NE}} - \boldsymbol{x}^{(0)}\right\|_2 \ ,$$

where $\eta = 1 - \frac{A^2}{NL^2}$.

It remains an open question to know if strong stability is also sufficient to ensure the convergence of the Best Response iterates (Algorithm 1). However, one particular case for which we have a convergence result is when costs functions are affine: $c_t = x \mapsto \alpha_t + \beta_t x$ with $\alpha_t \geq 0, \beta_t > 0$. In this case, the game is A-strongly stable with $A \stackrel{\text{def}}{=} 2\min_t \beta_t$, and we have:

Theorem 2 *Assume that, for each t, there exists $\alpha_t \geq 0, \beta_t > 0$ such that $c_t : x \mapsto \alpha_t + \beta_t x$. Then the sequence of iterates of Algorithm CBRD $\left(\boldsymbol{x}^{(k)}\right)_{k\geq 0}$ converges to the unique* NE $\boldsymbol{x}^{\mathrm{NE}}$ *of $\mathcal{G}$. Moreover, the convergence rate satisfies:*

$$\left\| \boldsymbol{x}^{\mathrm{NE}} - \boldsymbol{x}^{(k)} \right\|_2 \leq C \frac{\sqrt{L}N}{\sqrt{A}} \times \frac{1}{\sqrt{k}} \ ,$$

where C depends on $x^{(0)}$ and the billing functions, $L = 2\max_t \beta_t$ and $A = 2\min_t \beta_t$.

The proof relies on the fact that, in this case, the game has the potential property, and the Best Response algorithm is equivalent to an alternating block-coordinate minimization [1] on the potential function.

Application to Energy Management with Demand Response. We consider a set of electricity consumers $\mathcal{N}$ who are linked to a local aggregator. The aggregator wants to minimize the costs induced by the consumption profile on the set of time periods $\mathcal{T} = \{1, \dots, T\}$ and, for that, he sends to each consumer the tuple of per-unit energy price functions $(c_t(.))_t$. Each consumer i has some flexible electrical appliances (e.g. electric vehicle) and has a total flexible energy need E_i to be satisfied over $\mathcal{T}$, which gives the strategy set (feasible profiles):

$$\mathcal{X}_i \stackrel{\text{def}}{=} \{\boldsymbol{x}_i \in \mathbb{R}^T \ ; \ \sum_t x_{i,t} = E_i \text{ and } \forall t,\ \underline{x}_{i,t} \leq x_{i,t} \leq \overline{x}_{i,t}\} \ , \tag{3}$$

where $\overline{x}_{i,t}, \underline{x}_{i,t}$ are upper and lower bounds on the power i can asks on time period t. The total cost (energy bill) for consumer i is $b_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) = \sum_t x_{i,t} c_t(x_t)$. With the specific strategies set given by (3) and affine price functions (c_t), each iteration of Algorithm 1 or Algorithm 2 can be computed in $\mathcal{O}(T)$. Therefore, those algorithms can be efficiently used to compute the equilibrium consumption profile in a decentralized fashion and to coordinate consumers.

Références

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