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About prior saturation points for affine control systems[1]

Térence BAYEN
IMAG, Univ Montpellier, CNRS, Montpellier, France.

Olivier COTS
UMR CNRS 5505, IMA, ENSEIHHT Toulouse

Résumé. We consider minimal time control problems governed by an affine system w.r.t. the control. Our aim is to study properties of the optimal synthesis in presence of a singular locus that involves a saturation point for the singular control. We show that the optimal synthesis exhibits a prior saturation point at the intersection of the singular arc and a switching curve, and we also discuss qualitative properties of this curve. We highlight this phenomenon on several models arising in nuclear magnetic resonance and in bioprocesses.

Mots-clefs : Optimal control, Singular arc, Saturation phenomenon, Contrast imaging, Bioprocesses.

Minimal time control problems for affine systems with one input such as:

$$\dot{x} = f(x) + ug(x), \quad x \in \mathbb{R}^n, \quad |u| \leq 1, \tag{1}$$

have been investigated a lot in the literature, see *e.g* [2, 8, 12], [5] for $n = 2$, and references herein. One often encounters singular trajectories which appear when the switching function of the system is vanishing on a time interval $I := [t_1, t_2]$. In order to find an issue to a minimal time control problem governed by (1), one usually requires that the singular control u_s is admissible and non saturating, which means that

$$|u_s| < 1, \tag{2}$$

over I. This allows the trajectory to stay on the singular arc. However, one cannot in general show that this assumption holds. In fact, the singular control can be expressed as a function of the state x and adjoint state λ by

$$u_s(t) := -\frac{\langle \lambda(t), [f, [f, g]](x(t)) \rangle}{\langle \lambda(t), [g, [f, g]](x(t)) \rangle}, \quad t \in I, \tag{3}$$

and this expression does not always guarantee that (2) is satisfied. One can argue that it is enough to consider a larger admissible upper bound for the controls, but this seems rather artificial, and not necessarily feasible from a practical point of view (see for instance [10] where this situation is encountered). Our aim is to study properties of minimal time control problems in the plane where the singular control satisfies (2) only on a sub-domain of the state space (the part of the singular arc where (2) does not hold is usually called *barrier* [5, 8]). This may happen in many engineering problems in particular when the singular control can take

[1] This communication is based on the papers [1, 4] and on a work in progress by the authors.

arbitrary large values in the state space (see *e.g.* [1, 6, 10]). When $u_s(t) > 1$ for some instant $t \in I$, the system exhibits a *saturation phenomenon*. We then say that x^* is a saturation point if x^* is a point of the singular arc such that $u_s(t^*) = 1$ where $t^* \in (t_1, t_2)$ and such that $|u_s(t)| > 1$, resp. $|u_s(t)| < 1$, for $t \in (t^*, t_2]$, resp. for $t \in [t_1, t^*)$. Next, we suppose that such point exists[2] and that the singular arc is a *turnpike* (*i.e.* it is locally optimal, see [5]).

Thanks to Pontryagin's Principle [9], we show that singular optimal trajectories necessarily leave the singular arc at a *frame point* $\hat{x}$ called *prior saturation point* before reaching x^* (see [5] for a description of frame points). As a consequence, a singular extremal trajectory ceases to be optimal before reaching the saturation point (note that singular extremal trajectories are admissible until x^*). This rather non-intuitive phenomenon is also studied in [11] where local results are given.

Next, we analyze the behavior of optimal trajectories at the point $\hat{x}$. At this point, we show that singular trajectories necessarily switch to the maximal value for the control. Since the singular arc is of turnpike type, there must exist a switching curve $\mathcal{C}$ which emanates from the prior saturation point $\hat{x}$. The behavior of optimal trajectories at this point is interesting for the optimal synthesis (see [1]). Next, we discuss about the tangency of the dynamics with $u = 1$ at the prior saturation point $\hat{x}$ to the switching curve at this point. Such property involves the behavior of the input-output mapping on a singular arc in the plane. It is interesting for depicting optimal trajectories and it asserts that singular trajectories leave the singular locus tangentially to $\mathcal{C}$ at $\hat{x}$.

Finally, we provide two examples where such phenomenon appears. The first one describes a fed-batch bioreactor with one species, one substrate, and inhibition on the substrate (see [1]). Such process is widely used in waste water treatment industries, and the mathematical formulation was given in [7]. We show that if the volume of water to be treated is above a given threshold, then the saturation phenomenon appears implying the existence of a prior-saturation point. A complete optimal synthesis can be then given in this case (see [1]). We also depict an example of the prior saturation phenomenon for the minimal time saturation problem in medical imaging (see [3, 4]). The goal of the contrast imaging problem is to bring the magnetization vector towards the center of the Bloch ball in minimum time. This problem is related to the contrast imaging problem in nuclear magnetic resonance (see [3, 4]). These examples highlight the notion of *bridge* which plays an important role in these studies.

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[2] Note that one also could consider the case where $u_s(t^*) = -1$.

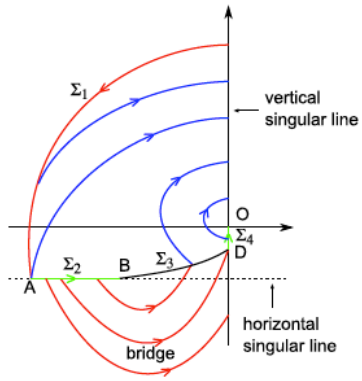

Figure 1: *Schematic representation of the optimal synthesis when the initial point of the dynamics is the north pole. Regular curves are plotted in blue (dark) and red (dark gray) for control fields equal to* $+1$ *and* -1*, respectively. The optimal singular trajectories are displayed in green (light gray). The black line is the switching curve, while the dashed one is the non-admissible part of the horizontal singular line. The point* B *is a saturation point and the point* D *is prior-saturation point.*

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