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Local convergence property of a regularized primal-dual method for bound constrained optimization

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Résumé. We present a primal-dual algorithm for solving a bound constrained optimization problem. The algorithm uses a regularization technique to handle the case where the second order sufficient optimality conditions do not hold at a local minimum. The local convergence analysis is done under a weaker assumption and is related to a local error bound condition. It is proved that locally the algorithm is superlinearly convergent. Some examples are also given to see the advantage of this new algorithm.

Mots-clefs : primal-dual method, interior point method, regularization, singular solutions, error bounds.

1 Introduction

We consider the bound constrained optimization problem

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t.} \quad & x \geq 0, \end{aligned} \tag{1}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a twice differentiable function. The first order optimality conditions can be written as

$$F(w) = 0 \quad \text{and} \quad w := (x, z) \geq 0,$$

where $F : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is defined by

$$F(w) = \begin{pmatrix} \nabla f(x) - z \\ XZe \end{pmatrix},$$

where $X = \operatorname{diag}(x_1, ..., x_n)$, $Z = \operatorname{diag}(z_1, ..., z_n)$ and $e = (1, \dots, 1)^\top$.
In order to solve the problem (1), we introduce a regularized primal-dual interior point algorithm, by considering a sequence of problems of the form

$$\min_{\substack{x \in \mathbb{R}^n \\ x > 0}} f(x) + \frac{\theta_k}{2} \|x - x_k\|^2 - \mu_k \sum_{i=1}^{n} \log[x]_i,$$

where x_k is the iterate at iteration k, μ_k is the barrier parameters and θ_k is the regularization parameter. The method is related to a proximal regularization scheme, see [2] and the references therein.

At each iteration, a Newton iterate $w_k^+ = (x_k^+, z_k^+)$ is computed by solving the linear system

$$\begin{pmatrix} H_k + \theta_k I & -I \\ Z_k & X_k \end{pmatrix} \begin{pmatrix} x_k^+ - x_k \\ z_k^+ - z_k \end{pmatrix} = - \begin{pmatrix} \nabla f(x_k) - z_k \\ X_k Z_k e - \mu_k e \end{pmatrix}, \tag{2}$$

where $w_k = (x_k, z_k)$ is the current iterate, $H_k = \nabla^2 f(x_k) + \delta_k I$, with $\delta_k \geq 0$. The parameter δ_k is chosen such that $H_k + X_k^{-1} Z_k \succeq 0$ to guarantee a descent property for the minimization algorithm. To maintain the strict feasibility of the iterates, the *fraction to the boundary rule* is applied: compute a step-length α_k as the greatest $\alpha \in (0, 1]$ such that

$$w_k + \alpha(w_k^+ - w_k) \geq (1 - \tau_k) w_k,$$

where $\tau_k \in (0, 1)$. The new iterate is then set according to

$$w_{k+1} := w_k + \alpha_k (w_k^+ - w_k).$$

The main purpose of this paper is to investigate the rate of convergence of this local algorithm without the strong second order sufficient condition. This one is replaced by a milder assumption related to an error bound condition.

2 Main result

The local convergence analysis is done under the assumptions of smoothness of the function f and of strict complementarity at a solution $w^* = (x^*, z^*)$. Contrary to the classical analysis, we do not assume that the second order sufficiency conditions (SOSC) hold at w^*. The SOSC are replaced by the following assumption. We say that the Hadamard product $x \circ \nabla f(x)$ provides a local error bound at a minimum x^* of problem (1), if there exist positive constants α and r such that

$$d(x, \mathcal{S}) \leq \alpha \|x \circ \nabla f(x)\| \quad \text{for all } x \in B(x^*, r),$$

where $d(x, \mathcal{S}) = \min_{\xi \in \mathcal{S}} \|x - \xi\|$ is the distance of x to set of optimal solutions of problem (1). This error bound condition is original and more general than the one of Tseng [4]: There exist positive constants δ and κ such that

$$d(x, S) \leq \kappa \| \min\{x, \nabla f(x)\}\| \quad \text{whenever} \quad \| \min\{x, \nabla f(x)\}\| \leq \delta.$$

Our analysis is mainly based on a property of uniform boundedness of the inverse of the matrix of the linear system (2). We show that this new property generalizes the one of [1].
The regularization and barrier parameters are chosen according to the following rules:

$$\theta_k = \gamma_1 \|F(w_k)\|^\sigma, \quad \mu_k = \gamma_2 \min\{\|F(w_k)\|^{1+\sigma}, \mu_{k-1}\}$$

where $\gamma_1 > 0$ and $\gamma_2, \sigma \in (0, 1]$. The sequence $\{\tau_k\}$ is chosen such that

$$1 - \tau_k = O(\|F(w_k)\|).$$

We have the following result about the superlinear convergence of the sequence $\{w_k\}$.

Theorem 1 *Let Ω be the set of primal-dual solutions of the problem* (1)*. There exists $\varepsilon > 0$ such that if at an iteration k_0, $w_{k_0} \in B(w^*, \varepsilon)$, then for all $k \geq k_0$, the sequence $\{w_k\}$ converges superlinearly to a solution $\widetilde{w} \in \Omega$ with a rate $1 + \sigma$.*

Finally, we give some examples to show the superlinear convergence of the new algorithm whereas some well-established algorithms such as IPOPT[5] and SPDOPT [3] just converge linearly.

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