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A convex selection theorem with a non separable Banach space

Pascal GOURDEL
Paris School of Economics, Université Paris 1 Panthéon-Sorbonne, Centre d'Economie de la Sorbonne-CNRS, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13.

Nadia MÂAGLI
Paris School of Economics, Université Paris 1 Panthéon-Sorbonne, Centre d'Economie de la Sorbonne-CNRS, 106-112 Boulevard de l'Hôpital, 75647 Paris Cedex 13.

Résumé. In a previous paper [3], we showed that if X is a metric space and Y is a Banach space, then any lower semicontinuous correspondence $\varphi : X \to 2^Y$ with nonempty convex valued such that φ has either closed or finite dimensional images admits a selection. In a new paper [4], we extend this result to the case where X is Hausdorff paracompact and perfectly normal topological space. This allows to revisit the pioneer work of Michael [2]) and to show that such a property is a characterization of Hausdorff paracompact and perfectly normal topological space. As in [3], we use the concept of peeling for the points x such that $\varphi(x)$ has a finite dimension in order to build a lower semicontinuous correspondence such that $\overline{\psi_\eta(x)} \subset \mathrm{ri}(\varphi(x))$. In [4], additional techniques are used to encompass the absence of a metric structure.

Mots-clefs : barycentric coordinates, paracompact, perfectly normal, continuous selections, lower semicontinuous correspondence, closed valued correspondence, finite dimensional convex values, separable Banach spaces.

1 Michael's selection theorems (1956)

The definition of perfect normal space we use can be found in [2], the definition of paracompact space is classical [1]. Let us first recall Michael's results.

Theorem 1 [Closed-Convex valued selection theorem]
Let X be a T_1 topological space, the following properties are equivalent:

- X is paracompact
- If Y is a Banach space, then any lower semicontinuous correspondence $\varphi : X \to \mathbb{R}$ with nonempty closed convex values admits a continuous single valued selection.

Theorem 2 [Michael (1956)] Let X be a T_1 space, the following properties are equivalent:

- X is a perfectly normal space
- If $\varphi : X \to \mathbb{R}$ is a lsc correspondence with nonempty convex values, then φ admits a continuous single valued selection.

Those results are very important results in order to get the existence of a selection but they also provide alternative characterizations of some topological properties. The following Michael selection theorem [2] dedicated for non-closed valued correspondences is much more difficult to prove. The assumption on Y is reinforced by adding the separability.

Theorem 3 [Convex valued selection theorem] Let X be a perfectly normal space, Y a separable Banach space and $\varphi : X \to Y$ a lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional, or closed, or have an interior point, then φ admits a continuous single-valued selection.

First note in order to compare Michael's results that perfectly normality does not imply paracompactness nor the converse.

2 Paracompact perfect normal spaces

Let us focus on this class of space. First we recall a classical proposition then we show that this property is hereditary one.

Proposition 1 Let X be a T_1 space. If X is perfectly normal, then for any two disjoints closed subsets F and G, there exists a continuous function $f : X \to [0,1]$ such that $F = f^{-1}(\{0\})$ and $G = f^{-1}(\{1\})$.

Proposition 2 Let X be a T_1 space, we assume that X is perfectly normal and paracompact. Then any subset Z of X is both paracompact and perfectly normal.

3 The tools

In order to build selection, we used very often this classical lemma.

Lemma Let X and Y be two topological spaces and F a closed subset of X. Suppose that $\varphi : X \longrightarrow 2^Y$ is a lsc correspondence and $f : F \to Y$ is a continuous single-valued selection of the $\overline{\varphi}_{|F}$ (the closure). Then, the correspondence ψ given by

$$\psi(x) = \begin{cases} \{f(x)\} & \text{if } x \in F \\ \varphi(x) & \text{if } x \notin F \end{cases}$$

is also lsc.

The proof of our main result is based on the concept of "peeling" introduced in [3]. Let us first recall that the relative interior of a convex set C is a convex set of same dimension and that $\mathrm{ri}(\overline{C}) = \mathrm{ri}(C)$ and $\overline{\mathrm{ri}(C)} = \overline{C}$.

Definition 1 Let C be a nonempty finite dimensional subset of Y. We say that C' is a peeling of C of parameter $\rho \geq 0$ if (cf. Figure 1)

$$C' = \Gamma(C, \rho) := \left\{ y \in C \text{ such that } \overline{B}(y, \rho) \cap \mathrm{aff}(C) \subset \mathrm{ri}(C) \right\}.$$

Definition 2 Let η be a non negative real-valued function defined on X, and φ a correspondence from X to Y. We will say that the correspondence $\varphi_\eta : X \longrightarrow 2^Y$, is a peeling of φ of parameter η if for each $x \in X$, we have $\varphi_\eta(x) = \Gamma(\varphi(x), \eta(x))$.

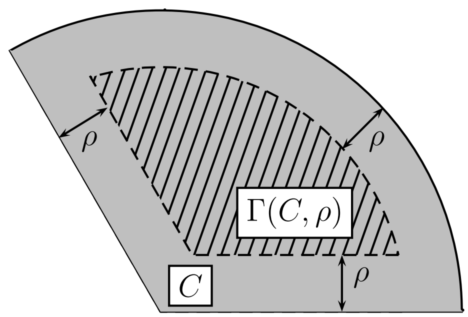

Figure 1: Peeling Concept

Proposition 3 Let X be a topological space, Y a Banach space and let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. Let D_i denotes $\{x \in X \mid \dim_a \varphi(x) = i\}$. If $\eta : D_i \to [0, +\infty[$ is continuous then φ_η is a lsc correspondance (possibly empty valued) with relatively open values.

Note that the above proposition does not hold true when there is a change of dimension of φ.

4 Our Results

Besides, as explained before, in his paper [2] (Example 6.3), Michael provided a counter example (a lsc correspondence φ from the closed unit interval $[0, 1]$ with nonempty open convex values of a Banach space Y for which there exists no selection). This shows that it is not possible to state the whole result without the assumption of separability.
The case where the correspondence is either finite dimensional or closed values has been studied in a previous paper but in a metric setting.

Theorem 4 [Gourdel-Mâagli (2016)] Let X be a metric space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional or closed, then φ admits a continuous single-valued selection h.

In order to provide a full answer to the case where the correspondence is either finite dimensional or closed values, we now state the main result of this paper.

Theorem 5 [4] Let X be a Hausdorff paracompact perfectly normal topological space and Y a Banach space. Let $\varphi : X \to 2^Y$ be a lsc correspondence with nonempty convex values. If for any $x \in X$, $\varphi(x)$ is either finite dimensional or closed, then φ admits a continuous single-valued selection h. Moreover, for any x in X, if $\varphi(x)$ has a finite dimension, then $h(x) \in \mathrm{ri}(\varphi(x))$.

It is worth noting that under the conditions of Theorem 5, we may have lsc correspondence with both convex closed and finite dimensional values, for example when the image is a singleton.

Corollary 1 Let X be a T_1 space, the following properties are equivalent:

- X is a paracompact and perfectly normal space
- If Y is a Banach space, any $\varphi : X \to Y$, which is a lsc correspondence with nonempty convex values such that $\varphi(x)$ is either finite dimensional or closed, admits a continuous single valued selection.

Références

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