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An efficient approach for solving large-scale non-convex quadratic problems

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Résumé. The mixed-integer quadratically-constrained quadratic programs are a very general class of instances that allow to model many real-life problems. We will present improvements on convex relaxations of such problems, and an augmented lagrangian dual approach to solve these relaxations. The framework is then integrated in a non-linear branch-and-bound algorithm to find near optimal bounds.

Mots-clefs : Mixed-integer quadratically-constrained quadratic programming, Nonconvex optimization, Augmented lagrangian, LocalSolver, Local search

The problem

We consider a general quadratic problem with quadratic constraints and mixed integer variables. Such a problem can be written

$$(QP) \quad \min_{l \leq x \leq u} x^T Q x + c^T x + \alpha$$
$$\text{s.t.} \begin{cases} x \in P, \\ x^T Q_i x + c_i^T x + \alpha_i \leq 0, \ i \in I, \\ x_j \in \mathbb{Z}, \ j \in J. \end{cases}$$

where P is a polyhedra. This problem is convex if and only if the quadratic expressions in the objective and constraints are convex, ie Q and the Q_i are positive semidefinite. We do not assume that this is the case.

We are interested in finding quasi-optimal solutions to (QP), meaning that we want a feasible solution and a lower bound whose associated optimality gap is of the order of magnitude of a few percent.

Existing methods

The method of choice for solving (QP) is the reformulation-linearization technique [1], that builds a linear relaxation of the problem and is then used in a branch and bound algorithm.

In the case of binary quadratic problems, several other methods have been widely used. Instead of a linear relaxation of the bilinear terms, a semidefinite formulation can be used [3], which leads to very tight relaxations at a more expensive cost.

Besides an automatic reformulation-linearization feature, many commercial QP solvers make use of the well-known diagonal shift method [4]. The idea is that the convex terms $x_i^2 - x_i$ cancel for a boolean solution, and a quadratic function $x^T Q x$ can be replaced by $x^T(Q + \text{diag}(\alpha))x - \alpha^T x$, where α is chosen such that the resulting matrix is positive semidefinite. A typical choice for α is the smallest (negative) eigenvalue of Q. Important improvements to this method have been proposed in [3], where an optimal α is computed using a semidefinite program. The resulting quadratic program is then solved with a standard QP solver. These improvements are refered to as the QCR method.

Our approach

While semidefinite programming is a powerful tool that leads to excellent bounds, it does not scale in the number of variables. On the other hand, the reformulation-linearization method is efficient on sparse problems only and needs a lot of branching to give good bounds. Primal feasible solutions are obtained with the local search solver LocalSolver [5], and we focus on the bounding procedure.

Our approach generalizes the method of the diagonal shift by using convexification functions of the form $\phi_\alpha(x) = \sum_i \alpha_i(x_i - l_i)(x_i - u_i)$. These convexification terms are non-positive on the domain $l \leq x \leq u$, and allow to generate a convex relaxation of the problem. The diagonal shift α is chosen with a heuristic approach that lies in between the uniform $\lambda_{\min}$ shift and the QCR method, with results close to the second at a cost close to the first.

The resulting relaxation is then solved with an iterative solver based on an augmented lagrangian algorithm. This allows for a lot of flexibility in the branch and bound and has some nice interactions with the convexification step, that can be carried at the root node only, or periodically.

Numerical benchmark against state-of-the-art solvers show that the approach is very promising and already improves the results on standard instances such as maxcut or k-cluster problems.

Références

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