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On Exact Polynomial Optimization

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Résumé. We consider the problem of finding exact sums of squares (SOS) decompositions for certain classes of non-negative multivariate polynomials, while relying on semidefinite programming (SDP) solvers.
We start by providing a hybrid symbolic-numeric algorithm computing exact rational SOS decompositions for polynomials lying in the interior of the SOS cone. This algorithm computes an approximate SOS decomposition for a perturbation of the input polynomial with an arbitrary-precision SDP solver. An exact SOS decomposition is obtained thanks to the perturbation terms. We prove that bit complexity estimates on output size and runtime are both polynomial in the degree of the input polynomial and simply exponential in the number of variables. This analysis is based on quantifier elimination as well as bounds on the cost of the ellipsoid method and Cholesky's decomposition.
Next, we apply this algorithm to compute exact Polya and Putinar's representations respectively for positive definite forms and polynomials positive over basic compact semialgebraic sets. We also compare the implementation of our algorithms with existing methods based on CAD or critical points.

Mots-clefs : optimization algorithms, positive polynomials, Polya's Positivstellensätz, Putinar's Positivstellensätz, sums of squares decomposition, semidefinite programming, real algebraic geometry.

Let $\mathbb{Q}$ (resp. $\mathbb{R}$) be the field of rational (resp. real) numbers. We consider the problem of deciding the non-negativity of $f \in \mathbb{Q}[\mathbf{x}]$ either over $\mathbb{R}^n$ or over a semi-algebraic set $\mathbf{K}$ defined by some constraints $g_1(\mathbf{x}) \geq 0, \dots, g_m(\mathbf{x}) \geq 0$ (with $g_j \in \mathbb{Q}[\mathbf{x}]$). Further, d will denote the maximum of the total degrees of these polynomials.
As many other algorithmic problems in effective real algebraic geometry, this one is known to be NP hard [13]. The famous Cylindrical Algebraic Decomposition algorithm [7] allows to solve such problems in time doubly exponential in n (and polynomial in d). This complexity result has been improved later on, through the so-called critical point method, starting from [8] and a series of works [16, 9] which culminates with [5] to establish that this decision problem can be solved in time $((s+1)d)^{O(n)}$ (see also [6, Chap. 13]). These latter ones have been extensively developed and optimized to obtain implementations which reflect the complexity gain (see e.g. [2, 1, 4, 3]). However, all in all, these techniques are singly exponential in n. Besides, these algorithms are "root finding" algorithms: to decide the positivity of f over the considered domain, one asks these algorithms to find a point therein at which f is negative. When f is positive, such algorithms will return an empty list without a *certificate* that can be checked *a posteriori*.

To bypass the curse of exponential algorithms while computing certificates of non-negativity for the problems considered here, an approach based on *sums of squares* (SOS) decompositions (and their variants) has been popularized by Lasserre [12] and Parillo [14]. We refer to [13] and references therein for detailed surveys on this approach. In a nutshell, the idea is as follows.
A polynomial f is non-negative over $\mathbb{R}^n$ if it can be written as an SOS $s_1^2 + \cdots + s_r^2$ with $s_i \in \mathbb{R}[\mathbf{x}]$ for $1 \leq i \leq r$. Also f is non-negative over the semi-algebraic set $\mathbf{K}$ if it can be written as $s_1^2 + \cdots + s_r^2 + \sum_{j=1}^m \sigma_j g_j$ where σ_i is a sum of squares in $\mathbb{R}[\mathbf{x}]$ for $1 \leq j \leq m$. It turns out that, thanks to the "Gram matrix method", following e.g. [12, 14], computing such decompositions can be reduced to solving Linear Matrix Inequalities (LMI), which boils down to considering a semidefinite programming (SDP) problem.
For instance, on input $f \in \mathbb{Q}[\mathbf{x}]$ of even degree $d = 2k$, the decomposition $f = s_1^2 + \cdots + s_r^2$ is a by-product of a decomposition of the form $f(\mathbf{x}) = v_k(\mathbf{x})^T L^T D L v_k(\mathbf{x})$ where v_k is the vector of all monomials of degree $\leq k$ in $\mathbb{Q}[\mathbf{x}]$, L is a lower triangular matrix with non-negative entries on the diagonal and D is a diagonal matrix with non-negative entries. The matrices L and D are obtained after computing a symmetric matrix G (the Gram matrix), semidefinite positive (all its eigenvalues are non-negative), such that $f = v_k^T G v_k$. Such a matrix G is found using solvers for LMIs. Even if such inequalities can be solved symbolically (see [10]), the degrees of the extensions are prohibitive on large examples. Besides, there exist fast numerical solvers for solving LMIs, e.g. SeDuMi [17], SDPA [18]. But using uniquely numerical solvers yields "approximate" non-negativity certificates. On our example, the matrices L and D (and consequently the polynomials $s_1, \ldots, s_r$) are not known exactly.
This raises topical questions. The first one is how to let interact symbolic computation with these numerical solvers to get *exact* certificates? What to do when SOS certificates do not exist? Also, given inputs with rational coefficients, can we obtain certificates with rational coefficients?
For these questions, we inherit from previous contributions from Parillo and Peyrl [15] and next Kaltofen, Li, Yang and Zhi [11].
This work provides a new algorithmic framework to handle (un)-constrained polynomial optimization problems with exact rational SOS decompositions. The first contribution is a hybrid symbolic-numeric algorithm, called `intsos`, providing rational SOS decompositions for polynomials belonging to the interior of the SOS cone. The main idea is to perturbate the input polynomial then to obtain an approximate Gram matrix of the perturbation by solving an SDP problem and eventually to recover an exact decomposition with the perturbation terms. Then, we rely on Algorithm `intsos` to compute SOS of rational functions for positive definite forms, based on Polya's representations, yielding a second algorithm, called `Polyasos`.
Finally, we rely on Algorithm `intsos` to compute weighted SOS decompositions for polynomials positive on basic compact semialgebraic sets, yielding a third algorithm, called `Putinarsos`. When the input is an n-variate polynomial of degree d with integer coefficients of maximum bitsize τ, we prove that Algorithm `intsos` runs in boolean time $\tau^2 d^{\mathcal{O}(n)}$ and outputs SOS polynomials of bitsize bounded by $\tau d^{\mathcal{O}(n)}$. This also yields bit complexity analysis for Algorithm `Polyasos` and Algorithm `Putinarsos`. To the best of our knowledge, this is the first complexity estimates for the output of algorithms providing exact multivariate SOS decompositions.
The three algorithms are implemented within a Maple library, called `multivsos`. We provide benchmarks evaluating the performance against existing methods based on CAD or critical points.

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