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Learning and convergence analysis in finite mean field games

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Abstract. In this presentation, we consider a finite version of Mean Field Games (MFGs) introduced by Gomes et al.[3] and study, first the convergence of the fictitious play procedure to equilibrium and second, the relation of some finite MFGs with the continuous first order MFGs system.

Keywords : optimal control, mean field games, fictitious play.

1 Introduction

In this article we consider a mean field game problem where the number of states and times are finite. This framework has been introduced by Gomes, Mohr and Souza in [3].
Our contribution to these type of games is twofold. First, we consider the *fictitious play* procedure. Loosely speaking, the procedure is that, at each iteration, a typical player implements a best response strategy to his *belief* on the action of the remaining players. The belief at iteration $n \in \mathbb{N}$ is given, by definition, by the average of outputs of decisions of the remaining players in the previous iterations $1, \ldots, n-1$. In the context of continuous MFGs, the study of the convergence of such procedure to an equilibrium has been first addressed in [2], for a particular class of MFGs called *potential MFGs*. This analysis has then been extended in [4], by assuming that the MFG is monotone.
Our second contribution concerns the relation between continuous and finite MFGs. We consider here a first order continuous MFG and we associate to it a family of finite MFGs defined on finite space/time grids. By applying the results in [3], we know that for any fixed space/time grid the associated finite MFG admits at least one solution. Moreover, any such solution induces a probability measure on the space of strategies. Letting the grid length tend to zero, we prove that the aforementioned sequence of probability measures is pre-compact and, hence, has at least one limit point. The main result of this article states that any such limit point is an equilibrium of the continuous MFG problem. To the best of our knowledge, this is the first result relating the equilibria for continuous MFGs, with the equilibria for finite MFGs, introduced in [3].

2 Model

Let $\mathcal{S}$ be finite and $\mathcal{T} = \{0, 1, 2, \cdots, m\}$, representing the set of states and time.

Definition 2.1 *We denote by $\mathcal{K}_{\mathcal{S},\mathcal{T}}$ the set of all maps $P : \mathcal{S} \times \mathcal{S} \times (\mathcal{T} \setminus \{m\}) \to [0, 1]$, called the transition kernels, such that $P(x, \cdot, k) \in \mathcal{P}(\mathcal{S})$ for all $x \in \mathcal{S}$ and $k \in \mathcal{T} \setminus \{m\}$, where $\mathcal{P}(\mathcal{S})$ represents the set of probability distributions over $\mathcal{S}$.*

The data of the problem consists of $M_0 \in \mathcal{P}(\mathcal{S})$, $c : \mathcal{S}\times\mathcal{S}\times\mathcal{P}(\mathcal{S})\times\mathcal{P}(\mathcal{S}) \to \mathbb{R}$ and $g : \mathcal{S}\times\mathcal{P}(\mathcal{S}) \to \mathbb{R}$, that are called respectively, the initial measure, the running and final cost functions. We call a tuple (U, M) with $U : \mathcal{T}\times\mathcal{S} \to \mathbb{R}$, $M : \mathcal{T} \to \mathcal{P}(\mathcal{S})$, an equilibrium solution to the finite MFG, if

$$
\begin{aligned}
(i) \quad & U(x,k) = \inf_{p\in\mathcal{P}(\mathcal{S})} \sum_{y\in\mathcal{S}} p_y\Big(c_{xy}(p, M(t_k)) + U(y, k+1)\Big), \\
(ii) \quad & M(x, k+1) = \sum_{y\in\mathcal{S}} M(y, t_k)\hat{P}(y, x, k), \\
(iii) \quad & U(\cdot, T) = g(\cdot, M(T)), M(0) = M_0
\end{aligned}
\tag{1}
$$

for an element $\hat{P} \in \mathcal{K}_{\mathcal{S},\mathcal{T}}$ such that $\hat{P}(x,\cdot,k)$ is an optimizer of problem (i)(1) for every $x \in \mathcal{S}, 0 \le k < m$. This formulation of equilibrium can be represented as well by following configuration. For $M : \mathcal{T} \to \mathcal{P}(\mathcal{S})$, define $J_M : \mathcal{K}_{\mathcal{S},\mathcal{T}} \to \mathbb{R}$ as

$$
J_M(P) := \sum_{k=0}^{m-1} \sum_{x,y\in\mathcal{S}} M_P^{M_0}(x,k)P(x,y,k)c_{xy}(P(x,k), M(k)) + \sum_{x\in\mathcal{S}} M_P^{M_0}(x,m)g(x, M(m)),
$$

where, for notational convenience, we have set $P(x,k) := P(x,\cdot,k) \in \mathcal{P}(\mathcal{S})$. We consider the following MFG problem: find $\hat{P} \in \mathcal{K}_{\mathcal{S},\mathcal{T}}$ such that

$$
\hat{P} \in \mathrm{argmin}_{P\in\mathcal{K}_{\mathcal{S},\mathcal{T}}}\ J_M(P) \quad \text{with} \quad M = M_{\hat{P}}. \tag{MFG$_d$}
$$

3 Fcititious play in finite MFG

The first question is how one can find an equilibrium. For this goal, we apply the fictitious play procedure by constructing $\{(P_n, M_n, \bar{M}_n)\}_{n\in\mathbb{N}}$ in the following recursive way: given $M_1 : \mathcal{T} \to \mathcal{P}(\mathcal{S})$ arbitrary, set $\bar{M}_1 = M_1$ and, for $n \ge 1$, define

$$
\begin{aligned}
P_n &:= \mathrm{argmin}_{P\in\mathcal{K}_{\mathcal{S},\mathcal{T}}}\ J_{\bar{M}_n}(P), \\
M_{n+1}(\cdot,k) &:= M_{P_n}^{M_0}(\cdot,k), \qquad \forall\, k = 0,\dots,m, \\
\bar{M}_{n+1}(\cdot,k) &:= \tfrac{n}{n+1}\bar{M}_n(\cdot,k) + \tfrac{1}{n+1}M_{n+1}(\cdot,k), \qquad \forall\, k = 0,\dots,m,
\end{aligned}
\tag{2}
$$

We proved the convergence of P_n to equilibrium, under monotonicity and suitable regularities assumptions.

4 Convergence analysis of finite MFG

Our second contribution, which can be considered independently from the previous result, is the convergence analysis of finite scheme (1) to first order MFG solution when discretization becomes finer. Let (N_n^s) and $\big(N_n^t\big)$ be two sequences of natural numbers such that $\lim_{n\to\infty} N_n^s = \lim_{n\to\infty} N_n^t = +\infty$ and let (ϵ_n) be a sequence of positive real numbers such that $\lim_{n\to\infty}\epsilon_n = 0$. Define $\Delta x_n := 1/N_n^s$ and $\Delta t_n := T/N_n^t$. For a fixed $n \in \mathbb{N}$, consider the discrete state set $\mathcal{S}_n$ and the discrete time set $\mathcal{T}_n$ defined as

$$
\begin{aligned}
\mathcal{S}_n &:= \big\{x_q := q\Delta x_n \ \mid\ q \in \mathbb{Z}^d,\ \ |q|_\infty \le (N_n^s)^2\big\} \subseteq \mathbb{R}^d, \\
\mathcal{T}_n &:= \big\{t_k := k\Delta t_n \ \mid\ k = 0,\dots,N_n^t\big\} \subseteq [0,T].
\end{aligned}
\tag{3}
$$

For this discretization, set the running cost,

$$c_{xy}(p, M) := \Delta t_n \left(\frac{1}{q} \left| \frac{y-x}{\Delta t_n} \right|^q + f(x, M) \right) + \epsilon_n \log(p_y),$$

for a fixed quantity $q > 1$. Let (U_n, M_n, P_n) be an equilibrium solution of this finite MFG. For every $n \in \mathbb{N}$, the transition probability P_n induces a measure ξ_n on set of trajectories that are piecewise affine and taking values in $\mathcal{S}_n$ on times $\mathcal{T}_n$.
Let us recall the definition of MFG equilibria in terms of measures over trajectories. Given $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ consider the following family of optimal control problems, parametrized by the initial condition,

$$\inf \left\{ \int_0^T \left[\frac{1}{q} |\dot{z}(t)|^q + f(z(t), m(t)) \right] \mathrm{d}t + g(z(T), m(T)) \;\middle|\; z \in W^{1,q}([0,T]; \mathbb{R}^d),\;\; z(0) = x \right\}, \quad x \in \mathbb{R}^d. \tag{4}$$

Definition 4.1 *We call $\xi^* \in \mathcal{P}(\Gamma), e_0 \sharp \xi^* = m_0$ a MFG equilibrium for (4) if $[0,T] \ni t \mapsto e_t \sharp \xi^*$ belongs to $C([0,T]; \mathcal{P}_1(\mathbb{R}^d))$ and ξ^*-almost every γ solves the optimal control problem in (4) with $x = \gamma(0)$ and $m(t) = e_t \sharp \xi^*$ for all $t \in [0,T]$.*

We will prove that the induced measures (ξ_n) takes the accumulation points on equilibria measures. We first prove the convergence of (U_n) with using following auxiliary functions

$$U^*(x,t) := \limsup_{\substack{n\to\infty \\ \mathcal{S}_n \ni y \to x \\ \mathcal{T}_n \ni s \to t}} U_n(y,s), \qquad U_*(x,t) := \liminf_{\substack{n\to\infty \\ \mathcal{S}_n \ni y \to x \\ \mathcal{T}_n \ni s \to t}} U_n(y,s) \qquad \forall\, x \in \mathbb{R}^d,\;\; t \in [0,T]. \tag{5}$$

Proposition 4.1 *Assume that, as $n \to \infty$, $N_n^t / N_n^s \to 0$ and $\epsilon_n = o\left(\frac{1}{N_n^t \log(N_n^s)} \right)$. Then for every accumulation point $m : [0,T] \to \mathcal{P}(\mathbb{R}^d)$ of measures (M_n) we have $U^* = U_* = u$, where*

$$u(x,t) = \inf \left\{ \int_t^T \left[\frac{1}{q} |\dot{z}(s)|^q + f(z(s), m(s)) \right] \mathrm{d}s + g(z(T), m(T)) \;\middle|\; z \in W^{1,q}([0,T]; \mathbb{R}^d),\;\; z(t) = x \right\}. \tag{6}$$

Our principal result for this work will be the following.

Theorem 4.1 *There exists at least one limit point ξ^* of (ξ_n), with respect to the narrow topology in $\mathcal{P}(\Gamma)$, and every such limit point is a MFG equilibrium for (4).*

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