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Metrics and harmonicity in the space of games

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Résumé. In this work, we generalize the orthogonal decomposition of games presented in [2] to several metrics induced in the space of games. We introduce a family of inner products in the space of finite games and using the Helmholtz-Hodge decomposition theorem, we establish an orthogonal decomposition of finite games into three components, which we refer to as the *potential*, *μ-harmonic* and *non-strategic* components. We prove that a completely mixed strategy profile, related to the selected inner product, appears as an equilibrium in the μ-harmonic component and that μ-harmonic games do not generically have pure Nash equilibria. We further show that in two-player μ-harmonic games, coincidence between the sets of correlated and mixed equilibria holds true and if players have equal number of actions, it then follows uniqueness of the equilibrium.

Résumé long: In the context of non-cooperative game theory, the class of finite games can be seen as a finite-dimensional vector space. Several approaches for decomposing a game into simpler games, which admit more flexible equilibrium analysis, have been proposed so far in the literature. Endowing the space of games with an inner product, it is possible to quantify distance between games with respect to the induced norm. As a consequence, establishing an orthogonal decomposition for an arbitrary game, we can provide characterizations of its approximate equilibrium set through the equilibria of its components (see [2]). Hence, it is interesting to search for decompositions into components with desirable static properties and tractable dynamics. For instance, the class of potential games admit a pure Nash equilibrium and adaptive processes converging to it. In order to define a novel solution concept, [6] deal with cooperation-related issues that emerge in games with strategic players and propose a decomposition that relies on identifying *zero-sum* and *common-interest*, i.e., games where players have identical payoffs at any action profile, components for a given game. In general, the common-interest component is a potential game, and it can be used to approximate a given game with a potential one. However, this approximation need not yield the closest potential game. [2] provide an example, where although the original game is potential, the zero-sum and common-interest game decomposition may lead to a potential game which is much farther than the closest potential game. [3] provide a *strategic* and *behavioral* decomposition of games with two actions for each player and highlight that certain solution concepts are determined by a game's strategic part or influenced by the behavioral portion. More recently, [4] study the space of games as a Hilbert space and prove several decomposition theorems for arbitrary games identifying components such as potential games and games that are *strategically equivalent to zero-sum games*. To that point, the authors define strategical equivalence according to [2], i.e., two games are strategical equivalent if they share identical payoff differences for any deviating player. They further extend their results to games with continuous strategy sets. Using this decomposition, they also derive a characterization of zero-sum equivalent games and provide an alternative proof for the well-known characterization of potential games given by [1].

For our purpose, the leading decomposition result appears in [2]. The authors establish an orthogonal decomposition of games with respect to a natural extension of the standard inner product, into the *potential*, *harmonic* and *non-strategic* components. The non-strategic component reflects games in which each player's payoff depends only on the actions taken by the rest of the players. The first component captures games where players' preferences are aligned with a common function (potential function) and thus, admit a pure Nash equilibrium. On the other hand, conflicting strategic interactions are represented in harmonic games. Well-known games such as rock-paper-scissors and matching pennies belong in this class of games. In harmonic games there is no pure equilibria and the *uniformly mixed strategy profile*, i.e., each player selects with equal probability any of his actions, appears as an equilibrium. Clearly potential and harmonic games admit distinct equilibrium properties. Moreover, the authors prove that in two-player harmonic games the sets of mixed and correlated equilibria coincide and if players have equal number of actions then, two-player harmonic games are zero-sum and further there is uniqueness of the equilibrium.

The key-point of this decomposition result lies in a novel flow-representation of the preference structure by associating to each finite game an undirected graph, where nodes represent the set of action profiles and edges are defined between nodes that differ in the action of a single player. This graph representation captures the strategic aspect of Nash equilibria. The payoff

differences for the deviating players along the edges define a flow on the game-graph. This formalization enables first to outline the fundamental characteristics in preferences that lead to potential games and further to use the *Helmholtz-Hodge decomposition theorem*. According to this tool, a flow on a graph is decomposed into the components of *globally consistent*, *locally consistent* but *globally inconsistent*, and *locally inconsistent*. The first component represents a gradient flow while the second corresponds to flows around global cycles. In other words, the first component corresponds to flows in the image of the *gradient operator* and the second one to flows in the kernel of the *Laplacian operator*. The locally inconsistent component reflects circulations around 3-cliques of the graph; they are flows that belong in the image of the adjoint of the *curl-operator*. An implementation of Hodge theory in the context of *Statistical Ranking* is presented in [5]; from raw ranking data, the authors define pairwise rankings, represented as edge flows on an appropriate graph. Then, they prove that every edge flow can be decomposed into three orthogonal components with respect to the standard inner product, where each one reflects distinct ranking properties.

In this paper, we first introduce a family of inner products indexed by μ, in the spaces of real-valued functions defined on the nodes of an undirected graph and the one of flows on the edges of the graph. Then, for any inner product, we study the Helmholtz-Hodge decomposition of the flows on this graph. Following [2], we associate to each finite game an undirected graph. Consecutively, a family of inner products is induced in the space of finite games and since harmonic flows belong in the kernel of the adjoint of the gradient operator, we provide now a generalized definition of harmonic games with respect to the selected inner product. In order to fix a representative of strategically equivalent games with respect to μ, we present *μ-normalized* games. Using an appropriate projection operator, it turns out an orthogonal decomposition of finite games into the μ-normalized and non-strategic components. Then, we apply the Helmholtz-Hodge decomposition to flows generated by the first component. At that point, let us mention that graphs associated to games appear 3-cliques that correspond to consecutive deviations of the same player and as a consequence all the flows generated by games belong in the kernel of the curl-operator. This implies that locally inconsistent flows generated by games are reduced to the 0-flow. We end up establishing an orthogonal decomposition of finite games with respect to the selected inner product, into the potential, *μ-harmonic* and non-strategic components. We further prove that a completely mixed strategy profile related to the selected inner product, appears as an equilibrium in the μ-harmonic component. Hence, it is possible to characterize the approximate equilibrium set of any finite game in terms of the completely mixed and pure equilibria of the closest μ-harmonic and potential games respectively. We finally generalize the results of [2] concerning two-player harmonic games by showing that in μ-harmonic games, correlated equilibria are Nash equilibria and in particular, if players have equal number of actions, it follows uniqueness of the equilibrium.

Mots-clefs : Helmholtz-Hodge decomposition, Laplacian operator, gradient operator, curl operator, decomposition of games, harmonic games

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