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A unified splitting algorithm for composite monotone inclusions

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Abstract. Operator splitting methods have been recently concerned with inclusions problems based on composite operators made of the sum of two monotone operators, one of them associated with a linear transformation. We analyze here a general and new splitting method which indeed splits both operator proximal steps, and avoids costly numerical algebra on the linear operator. The family of algorithms induced by our generalized setting includes known methods like Chambolle-Pock primal-dual algorithm and Shefi-Teboulle Proximal Alternate Direction Method of Multipliers. A convergence analysis is given and separable augmented Lagrangian algorithms are derived from our scheme to solve convex optimization problems with separable structure and coupling constraints.

Keywords : Splitting methods, Proximal decomposition

Operator splitting methods have been recently concerned with inclusions problems based on composite operators made of the sum of two monotone operators, one of them associated with a linear transformation. We analyze here a general and new splitting method which indeed splits both operator proximal steps, and avoids costly numerical algebra on the linear operator. The family of algorithms induced by our generalized setting includes known methods like Chambolle-Pock primal-dual algorithm and Shefi-Teboulle's Proximal Alternate Direction Method of Multipliers.
We focus here on the composite inclusion

$$0 \in S(x) + A^*T(Ax) \tag{1}$$

where $S : X \mapsto X$ and $T : Y \mapsto Y$ are maximal monotone operators and $A : X \mapsto Y$ is a linear transformation with adjoint operator A^*.
Composite models involving sums and compositions of linear and monotone operators are very common and still challenging problems like constrained separable convex optimization, composite variational inequalities, or in large scale data analysis and statistical learning. Corresponding optimization problems take the form :

$$\min_{(x,y)} [f(x) + g(y) \mid Ax - y = 0]$$

Following Condat [1], we first state a generalized over-relaxed and multidimensional scaled proximal step iteration (to find a solution of $0 \in T(x)$), based on the generalized resolvent

$J_P^T := (T+P)^{-1}P$ where P is a positive semidefinite matrix to get the following iteration

$$z^{k+1} \in \rho J_P^T(z^k) + (1-\rho)z^k$$

with a relaxation parameter $\rho \in (0,2)$. To apply it to the composite model, we take the convex-concave saddle-point formulation so that our operator is

$$L(x,y,v) = \begin{pmatrix} \partial f(x) \\ \partial g(y) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & A^T \\ 0 & 0 & B^T \\ -A & -B & 0 \end{pmatrix} \begin{pmatrix} \partial x \\ \partial y \\ v \end{pmatrix}$$

We then choose a special matrix P in order to provide a separable structure to the algorithm :

$$P := \begin{bmatrix} C_1 & (\gamma^2-1)A^T MB & \gamma A^T \\ (\gamma^2-1)B^T MA & C_2 & \gamma B^T \\ \gamma A & \gamma B & M^{-1} \end{bmatrix}$$

for C_1, C_2 symmetric and M symmetric positive definite.
The corresponding algorithm takes the following form (taking $\rho = 1$ to simplify):

$$\left\{ \begin{array}{lcl} x^{k+1} & = & \mathrm{argmin}_x\, f(x) + \frac{1}{2}\|Ax + By^k + M^{-1}v^k\|_M^2 + \frac{1}{2}\|x - x^k\|_{M_1}^2 \\ y^{k+1} & = & \mathrm{argmin}_y g(y) + \frac{1}{2}\|Ax^{k+1} + By + M^{-1}v^k\|_M^2 + \frac{1}{2}\|y - y^k\|_{M_2}^2 \\ v^{k+1} & = & v^k + M(Ax^{k+1} + By^{k+1}) \end{array} \right. \quad (2)$$

We show how this generalized splitting scheme includes as special cases Chambolle and Pock's primal-dual splitting [2] and the Proximal ADMM algorithms given by Shefi and Teboulle [3]. Finally, we analyze some new special cases where the choice of P induces fixed-point iterations with some co-coercive maps and we prove the convergence to that fixed point. Thus the generalized splitting appears as a primal-dual generalization of Douglas-Rachford splitting (see [4]). Ergodic rates of convergence of the generalized splitting scheme are then obtained.

Références

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