Necessary Conditions and Lipschitz Continuity of the Value Function for Infinite Horizon optimal control problems under State Constraints

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Résumé. In this talk I will discuss sufficient conditions for Lipschitz regularity of the value function for an infinite horizon optimal control problem subject to state constraints. I focus on problems with cost functional admitting a discount rate factor and allow time dependent dynamics and lagrangian. Furthermore, state constraints may be unbounded and may have a nonsmooth boundary. Lipschitz regularity is recovered as a consequence of estimates on the distance of a given trajectory of control system from the set of all its viable (feasible) trajectories, provided the discount rate is sufficiently large. As the first application it is shown that the value function of the original problem coincides with the value function of the relaxed infinite horizon problem. The second application concerns first order necessary optimality conditions: a constrained maximum principle and sensitivity relations involving generalized gradients of the value function.

Mots-clefs : Optimal control, infinite horizon, state constraints.

Consider the infinite horizon optimal control problem $\mathscr{B}_\infty$

$$\text{minimize} \int_{t_0}^{\infty} e^{-\lambda t} l(t, x(t), u(t))\, dt \tag{1}$$

over all trajectory-control pairs $(x(\cdot), u(\cdot))$ subject to the state constrained control system

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [t_0, \infty) \\ x(t_0) = x_0 & \\ u(t) \in U(t) & \text{a.e. } t \in [t_0, \infty) \\ x(t) \in A & \forall\, t \in [t_0, \infty) \end{cases} \tag{2}$$

where $\lambda > 0$, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $l : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given functions, $U : [0, \infty) \rightrightarrows \mathbb{R}^m$ is a Lebesgue measurable set-valued map with closed nonempty images, A is a closed subset of $\mathbb{R}^n$, and $(t_0, x_0) \in [0, \infty) \times A$ is the initial datum. Every trajectory-control pair $(x(\cdot), u(\cdot))$ that satisfies the state constrained control system (2) is called *feasible*. The infimum of the cost functional in (1) over all feasible trajectory-control pairs, with the initial datum (t_0, x_0), is denoted by $V(t_0, x_0)$ (if no feasible trajectory-control pair exists at (t_0, x_0) or if the integral in (1) is not defined for every feasible pair, then, by definition, $V(t_0, x_0) = +\infty$). The function $V : [0, \infty) \times A \to \mathbb{R} \cup \{\pm\infty\}$ is called the *value function* of problem $\mathscr{B}_\infty$.
Infinite horizon problems have a very natural application in mathematical economics (see, for instance, the Ramsey model in [5]). In this case the planner seeks to find a solution to $\mathscr{B}_\infty$ (dealing with a maximization problem instead of a minimization one) with $L(t, x, u) =$

$e^{-\lambda t}l(ug(x))$ and $f(t,x,u) = \tilde{f}(x) - ug(x)$, where $l(\cdot)$ is called the "utility" function, $\tilde{f}(\cdot)$ the "production" function, and $g(\cdot)$ the "consumption" function, while the variable x stands for the "capital" (in some applications one takes as constraint set $A = [0,\infty)$ with $U(\cdot) \equiv [-1,1]$, but, in general, $x(\cdot)$ is a vector valued function and A is the cone of positive vectors).
It happens quite often, in mathematical economics papers, that one considers as candidates for optimal solutions only trajectories satisfying simultaneously the unconstrained Pontryagin maximum principle and the state constraints. Such an approach, however, is incorrect as there are cases where no optimal trajectory belongs to this class. There is, therefore, the need of a constrained maximum principle for infinite horizon problems with sufficiently general structure. The literature dealing with necessary optimality conditions for unconstrained infinite horizon optimal control problems is quite rich (see, e.g., [6] and the reference therein), mostly under assumptions on f and L that guarantee the Lipschitz regularity of $V(\cdot,\cdot)$. On the contrary, recovering optimality conditions in the presence of state constraints appears quite a challenging issue for infinite horizon problems, despite all the available results for constrained Bolza problems with finite horizon (cfr. [7]).
In the work [1], the normal maximum principle together with partial and full sensitivity relations and a transversality condition at the initial time are proved, under mild assumption on dynamics and constraints. To describe our results, assume $V(t,\cdot)$ is locally Lipschitz and denote by $N_A(y)$ the limiting normal cone to A at y. If $(\bar{x},\bar{u})$ is optimal for $\mathscr{B}_\infty$ at (t_0,x_0), then it was shown in [1] that there exists a locally absolutely continuous co-state $p(\cdot)$, a nonnegative Borel measure μ on $[t_0,\infty)$, and a Borel measurable selection $\nu(\cdot) \in \overline{\text{co}}\, N_A(\bar{x}(\cdot)) \cap \mathbb{B}$ such that $p(\cdot)$ satisfies the adjoint equation

$$-p'(t) \in \partial_x f(t,\bar{x}(t),\bar{u}(t))\,(p(t)+\eta(t)) - e^{-\lambda t}\partial_x l(t,\bar{x}(t),\bar{u}(t)) \qquad \text{a.e. } t \in [t_0,\infty),$$

the maximality condition

$$\begin{aligned}&\langle\, p(t)+\eta(t), f(t,\bar{x}(t),\bar{u}(t))\rangle - e^{-\lambda t}l(t,\bar{x}(t),\bar{u}(t))\\ &= \max_{u\in U(t)}\Big\{\langle\, p(t)+\eta(t), f(t,\bar{x}(t),u)\rangle - e^{-\lambda t}l(t,\bar{x}(t),u)\Big\} \qquad \text{a.e. } t \in [t_0,\infty),\end{aligned}$$

and the transversality and sensitivity relations

$$-p(t_0) \in \partial_x V(t_0,\bar{x}(t_0)), \quad -(p(t)+\eta(t)) \in \partial_x V(t,\bar{x}(t)) \qquad \text{a.e. } t \in (t_0,\infty), \tag{3}$$

where $\eta(t_0) = 0$, $\eta(t) = \int_{[t_0,t]} \nu(s)\,d\mu(s)$ for all $t \in (t_0,\infty)$, $\partial_x V$ and $\partial_x l$ stands for the generalized gradient of $V(t,\cdot)$ and $l(t,\cdot,\bar{u}(t))$, and $\partial_x f$ for the generalized jacobian of $f(t,\cdot,\bar{u}(t))$. Observe that, if $\bar{x}(\cdot) \in \text{int}\, A$, then $\nu(\cdot) \equiv 0$ and the usual maximum principle holds true. But if $\bar{x}(t) \in \partial A$ for some time t, then a measure multiplier factor, $\int_{[0,t]} \nu\, d\mu$, may arise modifying the adjoint equation.
In the literature one finds some results concerning continuity of the value function for state constrained infinite horizon problems, see for instance [3]. However in this last reference the state constraints are given by a compact set with a smooth boundary. This clearly does not fit the state constraint described by the cone of positive vectors. In addition, results of [3] address only the autonomous case, which is also a serious restriction, because, as it was shown later on, arguments of its proof can not be extended to the non-autonomous case whenever the time dependence is merely continuous. Because of their presence in various applied models, addressing non-autonomous control systems subject to unbounded and non smooth state constraints remains crucial. Let us note that (the finite horizon) state-constrained Mayer's and

Bolza's problems have been successfully investigated by many authors. However in the infinite horizon framework these results can not be used, because restricting optimal trajectories of the infinite horizon problem to a finite time interval, in general, does not lead to optimal trajectories of the corresponding finite horizon problem (cfr. [4] for a further discussion).
Infinite horizon problems exhibit many phenomena not arising in the finite horizon context and for this reason their study is still going on (cfr. [6]). To justify the above necessary optimality conditions it remains to guarantee the Lipschitz continuity of $V(t,\cdot)$. This is an easy step for finite horizon problems. However in our case the distance between two trajectories corresponding to the same control and to distinct initial conditions growing exponentially, it may go to infinity.
When the discount factor is present and in the absence of state constraints, the Lipschitz continuity has been recovered by many authors whenever λ is sufficiently large (in order to control the growth of l along trajectory-control pairs), see for instance [4] and its bibliography. But in the presence of state constraints the question becomes much more complex and a uniform inward pointing condition has to be imposed on the boundary of A in addition to having large discount factor. In [2] we proposed sufficient conditions for Lipschitz regularity of $V(t,\cdot)$ allowing both f and l to be time dependent and not requiring neither boundedness of A nor smoothness of ∂A. Furthermore, the very same assumptions allow us to show that the value function of the original problem $\mathscr{B}_\infty$ coincides with the value function of the relaxed infinite horizon problem. This last question is very important for the existence of relaxed optimal controls.

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