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# UPPER AND LOWER COUPLING CONJUGACY INEQUALITIES WITH APPLICATION TO BELLMAN EQUATION

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**Résumé.** Given a coupling between "primal" and "dual" sets, we define *upper and lower coupling transformations*. These transformations are general in that they cover and generalize both Fenchel-Moreau conjugates and inf-convolution. Thus equipped, we prove a general implication that relates an inequality involving "primal" sets to a reverse inequality involving the "dual" sets as follows. Let be given four sets $\mathbb{C}$, $\mathbb{D}$, $\mathbb{C}^\sharp$, $\mathbb{D}^\sharp$, together with three coupling functions $\mathbb{C} \overset{\Phi}{\leftrightarrow} \mathbb{C}^\sharp$, $\mathbb{D} \overset{\Psi}{\leftrightarrow} \mathbb{D}^\sharp$ and $\mathbb{C} \overset{\mathcal{K}}{\leftrightarrow} \mathbb{D}$. Consider two functions $f$ and $g$ defined respectively on "primal" sets $\mathbb{C}$ and $\mathbb{D}$. We prove that $f(c) \geq \inf_{d \in \mathbb{D}} \Big( \mathcal{K}(c,d) \dotplus g(d) \Big)$ $\Rightarrow f^{\Phi}(c^\sharp) \leq \inf_{d^\sharp \in \mathbb{D}^\sharp} \Big( \mathcal{K}^{\Phi \dotminus \Psi}(c^\sharp, d^\sharp) \dotplus g^{\Psi}(d^\sharp) \Big)$, where we stress that the Fenchel-Moreau conjugates are not necessarily taken with the same coupling. As an application, we consider the Bellman equation in stochastic dynamic programming. We prove that the Bellman operator is a coupling transformation for a proper Hamiltonian coupling. Then, we provide a "Bellman-like" equation for the Fenchel-Moreau conjugates of the value functions, the conjugacy being for whatever couplings.

**Mots-clefs :** couplings, Fenchel-Moreau conjugacy, Bellman equation

## Références

[1] J.-J. Moreau. Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. Pures Appl. (9)*, 49:109–154, 1970.