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SEMICONVEXITY OF THE BILATERAL MINIMAL TIME FUNCTION: THE NONLINEAR CASE

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Résumé. Under suitable assumptions, the epigraph of the bilateral minimal function is proved to be φ-convex for a nonlinear control system. This generalized, to the nonlinear case, the main result of [4] where a similar result is proved for a linear control system.

Mots-clefs : Bilateral minimal time function, linear and nonlinear control system, semiconvexity, φ-convexity, proximal analysis, nonsmooth analysis.

We consider a control system governed by a differential inclusion. Let F be a multifunction mappings points x in $\mathbb{R}^n$ to subsets $\mathbb{R}^n$. Associated with F the differential inclusion:

$$\dot{x}(t) \in F(x(t)) \text{ a.e. } t \in [0,T],\ x(0) = x_0. \tag{1}$$

A solution to (1) is an absolutely continuous function $x(\cdot)$ defined on the interval $[0,T]$ with initial value $x(0) = x_0$, in which case we say that $x(\cdot)$ is a *trajectory* of F originating from x_0. The *bilateral minimal time function* $T : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow [0,+\infty]$ associated to (1) can be defined as follows: For $(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n$, $T(\alpha,\beta)$ is the minimum time taken by a trajectory to go from α to β. When no such trajectory exists, $T(\alpha,\beta)$ is taken to be $+\infty$. The effective domain of $T(\cdot,\cdot)$ is denoted by $\mathcal{R}$, that is,

$$\mathcal{R} := \{(\alpha,\beta) \in \mathbb{R}^n \times \mathbb{R}^n : T(\alpha,\beta) < +\infty\}.$$

The function $T(\cdot,\cdot)$ was introduced in Clarke and Nour [1], see also [3], where the Hamilton-Jacobi equation of the time-optimal control problem was studied in a domain which contains the target set. These authors used $T(\cdot,\cdot)$ in order to construct proximal solutions to this equation, and to the study the existence of time-semigeodesic trajectories. The regularity of $T(\cdot,\cdot)$ was studied by Nour in [2], where necessary and sufficient conditions were provided for $T(\cdot,\cdot)$ to be continuous and to be locally Lipschitz in $\mathcal{R}$. More precisely, it was proven that $\mathcal{R}$ is open and $T(\cdot,\cdot)$ is continuous in $\mathcal{R}$ if and only if $T(\cdot,\cdot)$ is continuous at (α,α) for all $\alpha \in \mathbb{R}^n$, which is in turn equivalent to F and $-F$ to be *small-time locally controllable* at α for all $\alpha \in \mathbb{R}^n$. Since the small-time controllability of F and $-F$ at α, for all $\alpha \in \mathbb{R}^n$, is a quite strong hypothesis, in [2, Proposition 4.2] a sufficient condition for *local* continuity was provided as well. In particular, for $(\alpha,\beta) \in \mathcal{R}$, if F and $-F$ are small-time locally controllable at α *or* at β then $T(\cdot,\cdot)$ is continuous at (α,β). For the Lipschitz continuity, Nour proved in [2] that for $(\alpha,\beta) \in \mathcal{R}$, if $0 \in \operatorname{int} F(\alpha)$ *or* $0 \in \operatorname{int} F(\beta)$ then $T(\cdot,\cdot)$ is Lipschitz near (α,β). As a consequence, Nour deduced in [2, Proposition 4.6] that $\mathcal{R}$ is open and $T(\cdot,\cdot)$ is locally Lipschitz in $\mathcal{R}$ if and only if $0 \in \operatorname{int} F(\alpha)$ for all $\alpha \in \mathbb{R}^n$. Another useful result in [2] is the following theorem, concerning local *semiconvexity* of the bilateral minimal time function. By the linearity of F, we mean that $F(x) := Ax + U$ for all $x \in \mathbb{R}^n$, where A is an $n \times n$ matrix and U is a convex and compact subset of $\mathbb{R}^n$.

Theorem 1 ([2, Corollary 4.8]) *Assume F is linear and that $0 \in \operatorname{int} F(\alpha)$ or $0 \in \operatorname{int} F(\beta)$, where $(\alpha, \beta) \in \mathcal{R}$ with $\alpha \neq \beta$. Then $T(\cdot, \cdot)$ is semiconvex near (α, β) (that is, on an open set containing (α, β)).*

We recall that a function $f : U \longrightarrow \mathbb{R}^n$ is said to be semiconvex on $U \subset \mathbb{R}^n$ if for any convex $C \subset\subset U$ there exists $K_C > 0$ such that the function $x \mapsto f(x) + K_C\|x\|^2$ is convex on C, where $\|\cdot\|$ denotes the euclidean norm. The semiconvexity property can be seen as an intermediate property between Lipschitz continuity and continuous differentiability. More precisely, semiconvex functions are essentially a quadratic perturbation of convex functions and therefore inherit several regularity properties from convexity such as local Lipschitzianity and a.e. twice differentiability in the interior of their domain. Moreover, their epigraphs satisfy an external sphere condition with locally uniform radius; this property, for general sets, is often referred to *positive reach*, *proximal smoothness* and *φ-convexity*. Such functions are semiconvex if and only if they are locally Lipschitz and then are good candidate to extend Theorem 1 under a weaker condition. This was done in [4] as the following.

Theorem 2 ([4, Theorem 1.2]) *Assume F is linear and that F and $-F$ are small-time locally controllable near α or β, where $(\alpha, \beta) \in \mathcal{R}$ with $\alpha \neq \beta$. Then the epigraph of $T(\cdot, \cdot)$ is φ_0-convex near (α, β).*

The goal of this work is to generalize the preceding theorem to a nonlinear control system. More precisely, under the following three assumptions on F, we will prove that the epigraph of the function $T(\cdot, \cdot)$ is locally φ-convex:

- Standing hypotheses ($F(x)$ is nonempty, convex and compact, F is upper semnicontinuous and the linear growth condition is satisfied).
- Assumptions on the maximized Hamiltonian H associated with F.
- A φ_0-convexity assumption on the reachable set.

Références

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